

Time, Distance, Velocity, Redshift: a personal guided tour

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Abstract An attempt to answer the question “Can we observe galaxies that recede faster than light” led to a re-examination of the notions of time, distance, velocity and redshift as they occur in newtonian physics, special relativity, general relativity and cosmology. A number of misconceptions were uncovered. It was found that, once freed of preconceptions of special relativity the above question is easily and unequivocally answered.

Over the past fifteen years or so I have been a student of cosmology. I was puzzled by the following question: according to the celebrated Hubble’s Law that says velocity of recession is proportional to distance, there must come a distance so large that a galaxy located there is receding at the speed of light. What then happens to the photon the galaxy emits in our direction ? Will that photon ever reach us ? Or if it does reach us, will its redshift be infinite and hence it will make zero impact ? In that case, that distance must be marking the boundary of our *observable* universe; is that distance what is known as horizon ? Or has the Hubble Law broken down long before that point ? Or have we missed the point entirely and forgotten that velocity at a distant point is simply not defined in General Relativity ? I eventually worked out the answers and these were given in my (1997) paper. And it dawned on me that all the above “questions” are wrong questions, prompted by what I call the “Shadows of Special Relativity”. Once it is recognized that modern cosmology, dubbed “relativistic”, is based *not* on Special Relativity, but on General Relativity, and *not* on General Relativity alone, *but also on* the Cosmological Principle, the answers emerge with no more than first-year college mathematics. The present essay is a systematic clearing-up of all the dead wood and misconceptions encountered on the way. I have not refrained from using non-technical language, homely analogies, formal comparisons whenever these contribute to understanding.

1. Event-Intervals and Observers

Physics deals with events. But not *individaul* events; rather, what matters is the *interval* between specified events. An *observer* O specifies a given event

by three (newtonian) spatial coordinates (x, y, z) , and one (newtonian) time coordinate t . Consider a second observer O' , fixed at (X_0, Y_0, Z_0) with respect to O , and whose spatial axes $O'x'y'z'$ are parallel to $Oxyz$. Let O' specify the same event by (x', y', z', t') . It is usually presumed that according to Newton we should have simply $t' = t$. But that is *not* the essence of newtonian time. Newton characterises time as that which by its own nature *flows* evenly without reference to anything else, which means that so long as the clock *rates* of O and O' are the same, their *readings* can differ by a constant amount, T_0 , say. For this event, then, we have

$$\left. \begin{aligned} x &= x' + X_0, & y &= y' + Y_0, & z &= z' + Z_0, \\ t &= t' + T_0. \end{aligned} \right\} \quad (1)$$

Now consider two events E_1 and E_2 , with their respective specifications by O and O' :

$$\left. \begin{aligned} E_1 &\equiv (x_1, y_1, z_1, t_1) \equiv (x'_1, y'_1, z'_1, t'_1), \\ E_2 &\equiv (x_2, y_2, z_2, t_2) \equiv (x'_2, y'_2, z'_2, t'_2). \end{aligned} \right\} \quad (2)$$

For each of the two events, relation (1) holds, hence, for the interval between the two events we have, writing Δx for $x_2 - x_1$ etc.,

$$\left. \begin{aligned} \Delta x &= \Delta x', & \Delta y &= \Delta y', & \Delta z &= \Delta z', \\ \Delta t &= \Delta t'. \end{aligned} \right\} \quad (3)$$

These equations are what Newton would have for two observers at rest each to the other.

What happens if O' has a velocity v relative to O along the x -axis? Note, so far, only newtonian concepts of time, distance and velocity are involved. Continuing within the newtonian framework, we say that in that case, we have

$$\left. \begin{aligned} \Delta x &= \Delta x' + v\Delta t', \\ \Delta t &= \Delta t'. \end{aligned} \right\} \quad (4)$$

and $\Delta y = \Delta y', \Delta z = \Delta z'$. From this point on, these two trivial relations will generally be left implicit. Note in the first of the two equations, Newton would have written the additional term due to v as $v\Delta t$ and left the second equation implicit: I have written out the second and used it to put the first in the form shown, in order to contrast it with what is forthcoming.

Eqs. (4), then, are the coordinate (difference) transformation according to Newton for two observers in relative (uniform) motion.

2. Special Relativity: Some Common Misconceptions

Now consider a photon moving in the x -, or equivalently, the x' -direction and let us identify the two events E_1 and E_2 with the photon occupying two specific points at two instants of time. Then, if we write c ($=\Delta x/\Delta t$) for its

speed measured by O, and c' ($= \Delta x' / \Delta t'$) for its speed measured by O', then by dividing the first equation in (4) by the second, we obtain

$$c = c' + v. \quad (5)$$

This is contrary to Galileo's principle that no experiments can detect a uniform motion. And Einstein was such a firm believer in the Galilean principle that he would rather have the transformation (4) changed so as to have $c = c'$ for all observers in relative (uniform) motion. The result is the famous Lorentz transformation, which is most neatly written when expressed in some $c = 1$ units (e.g., all time-intervals in years and all distances in light-years), thus:

$$\left. \begin{aligned} \Delta x &= \gamma \Delta x' + \beta \gamma \Delta t' \\ \Delta t &= \beta \gamma \Delta x' + \gamma \Delta t' \end{aligned} \right\} \quad (6)$$

where $\beta = v/c$, $\gamma = 1/\sqrt{1 - \beta^2}$.

From this point on, $c = 1$ units will always be assumed. Readers may have noticed that most textbooks write the Lorentz transformation differently, that they would drop the Δ 's and write x for Δx , etc., thus:

$$\left. \begin{aligned} x &= \gamma x' + \beta \gamma t' \\ t &= \beta \gamma x' + \gamma t' \end{aligned} \right\} \quad (7)$$

This form is what the form (6) reduces to in the special case where O and O' both assign the coordinate values (0, 0, 0, 0) to the event E₁. It is my belief that it is the failure to recognize that the form (7) represents a special case and is not *always* applicable that is the cause for all the confusion surrounding the so-called "twin paradox" (Kiang 1992).¹

It is interesting to compare the Lorentz transformation (6) with the Newton transformation (4). The spatial interval Δx is *qualitatively* the same in both (being dependent on both $\Delta x'$ and $\Delta t'$), but is *quantitatively* different (the coefficients are different), while the time interval Δt is even *qualitatively* different: in Newton, Δt simply equals $\Delta t'$; in Lorentz, Δt depends on both $\Delta t'$ and $\Delta x'$ (and in the same way as Δx depends on $\Delta x'$ and $\Delta t'$).

Eqs. (6) imply $(\Delta t)^2 - (\Delta x)^2 = (\Delta t')^2 - (\Delta x')^2$. Then, incorporating the two trivial relations spelled out above, we can subsume the whole family of Eqs. (6) for all values of v between 0 and 1 by one single statement, namely, the finite "spacetime interval" Δs defined by

$$(\Delta s)^2 = (\Delta t)^2 - [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2] \quad (8)$$

¹The differential ageing between two twins, one a stay-at-home, one a space traveller, can be illustrated with the following simple numbers. On their common 20th birthday, twin A stays behind while twin B goes out into space at speed 0.6c. On B's 30th birthday he turns round and heads back at the same speed 0.6c. He reaches home on his 40th birthday, just to find A celebrating *his* 45th birthday. Between parting and re-unison (two specific events), B has aged 20 years and A, 25 years. This result follows unequivocally from the form (6), when we recognize that, here, *three* inertial frames (or newtonian observers in relative (uniform) motion) are involved, No. 1 for A, No. 2 for B going out, No. 3 for B coming back. If we use the form (7) *both* for the transformation between No. 1 and No. 2 *and* for that between No. 1 and No. 3, then we could multiply paradoxes *ad infinitum*. For details, see Kiang (1992)

is invariant (has the same value) for all observers in relative (uniform) motion.

2.1 Minkowski noted that if we put $\tau = it$ in the above formula, then τ would be formally indistinguishable from x, y or z , so he wrote his famous words,

“Henceforth space by itself and time by itself, are doomed to fade away into mere shadows, and only a union of the two will preserve an independent reality” (quoted in Taylor and Wheeler 1963 p.37).

These words are sometimes taken (or mistaken) to mean that space and time must now be considered as indistinguishable. That time and space are indistinguishable is generally true in the next stage of the theoretical development—Einstein’s General Theory of Relativity (GR), but here at the stage of his Special Theory of Relativity (SR), it is not true. In SR, for a given observer, space intervals are still space intervals, time intervals are still time intervals: the observer conceives space and time exactly as Newton did. Only in SR, for two observers in relative (uniform) motion measuring the separation between the same pair of events, the time or spatial *measure* of one is *each* related to *both* the time and spatial *measures* of the other.

3. General Relativity: Merging of Time and Space

Gravity is not considered in SR. It was to incorporate gravity that Einstein developed his GR, with the guiding principle that GR reduces to SR at the local limit. In GR, an event is specified by four *generalised* coordinates, $(\xi^\mu, \mu = 0, 1, 2, 3)$ and, instead of *finite* intervals $\Delta\xi^\mu$, GR deals with *infinitesimal* intervals $d\xi^\mu$. Corresponding to (8), GR has as invariant the quantity, called “the metric” or the “line-element” ds , defined by

$$(ds)^2 = g_{\mu\nu} d\xi^\mu d\xi^\nu, \quad (9)$$

where a repeated index implies summation over $(0,1,2,3)$, and $g_{\mu\nu}$ ($g_{\mu\nu} = g_{\nu\mu}$ for $\mu \neq \nu$), are ten independent functions of ξ^μ , that inform the local gravitational field.

It is instructive to compare the two invariant forms, GR’s (9) and SR’s (8), or rather, the latter’s implied infinitesimal form,

$$(ds)^2 = (dt)^2 - [(dx)^2 + (dy)^2 + (dz)^2] \quad (10)$$

We see how much simpler (10) is: it contains no mixed terms such as “ $dt dx$ ” and the four non-zero $g_{\mu\nu}$ (particularly that factors $(dt)^2$) are all constants.

If in (10) we replace the rectangular coordinates (x, y, z) by the usual spherical coordinates (r, θ, ϕ) we get

$$(ds)^2 = (dt)^2 - \{(dr)^2 + r^2[(d\theta)^2 + \sin^2 \theta (d\phi)^2]\}. \quad (11)$$

This form of the Minkowski metric will be used below for comparison purposes.

3.1 Condition for a Universal Time in GR

At a given event, GR must reduce to SR, so at a given event, labelled 1, it is always possible for us to distinguish, say, ξ^0 as time, t . Can the same identification be made at another event 2 ? Now, in GR, this means we have to see what happens when we integrate ds between 1 and 2 along the path $\xi^1 = \xi^2 = \xi^3 = \text{const.}$, or $ds = \sqrt{g_{00}} d\xi^0$. Hence

- (i) if $g_{00} = \text{const.}$ at all events, then ξ^0 is indeed universal time; or,
- (ii) if at all events, g_{00} is a function only of ξ^0 , $f(\xi^0)$ say, then we can take $\int f(\xi^0) d\xi^0$ as universal time.

These conditions are so restrictive that, in the general case, we do not expect to be able to say which one of the four coordinates is time: time and space *are* mixed up in a single, four-dimensional continuum, spacetime.

Yet in one of the most important applications of GR, namely, cosmology, time and space separate out;—and in a most characteristic way too. The time that emerges is universal time and the space, is of a kind hitherto undreamed of.

4. Relativistic Cosmology: Space Evolves in Time

That this remarkable situation emerges in relativistic cosmology is because the latter is not just General Relativity: it is General Relativity plus the Cosmological Principle (CP), which says the universe is homogeneous and isotropic and looks the same to all observers.² The CP implies that the three-dimensional physical space must either be static, or expanding uniformly or contracting uniformly, in complete analogy with the corresponding behaviour of the two-dimensional surface of a balloon that is readily visualizable. Observations (mainly associated with the name of Edwin Hubble) then tell us that the universe is expanding, rather than static or contracting.

The solution of Einstein's field equation (the “master” equation of GR) under the CP is the Friedmann-Robertson-Walker (FRW) metric,

$$(ds)^2 = (dt)^2 - R^2(t)\{(dr)^2 + r^2[(d\theta)^2 + \sin^2 \theta (d\phi)^2]\}. \quad (12)$$

It should be said that this is only one (the simplest) of three possible forms of the FRW metric: it corresponds to the case of *flat* space ($k = 0$). If space is closed ($k = +1$) or open ($k = -1$), then the r^2 factoring the square brackets should be replaced by

$$ka_0^2 \sin^2(r/\sqrt{k}a_0), \quad (k = \pm 1),$$

containing an additional free parameter a_0 , the radius of curvature of space at the present epoch (Longair 1984, p.292). This way of specifying the FRW

²The last part is usually separately known as the Copernican Principle

metric has the merit that, irrespective of space curvature, r is always the (co-moving) radial coordinate itself;—and we shall be concerned exclusively with radial motions (e.g., Eq (17)).³

Let us now compare the FRW metric (12) with the “Minkowski” metric of SR, (11). Formally they have one feature in common; and there is also one all-important difference.

In both, we have a universal time t , as evidenced by the term $(dt)^2$ and by the absence of mixed terms such as $dt d\phi$. (cf. 3.1).

The difference is in what happens to the radial line element dr in the two cases. In the Minkowski, dr stands by itself; in the FRW, it is factored by $R(t)$, a function of the time t .

Recall that in SR, both time and space are newtonian (cf. 2.1). Hence we have that in relativistic cosmology, time is still newtonian,⁴ but space is no longer so: all spatial intervals now depend on a single, universal function of time, $R(t)$, called “scale factor”. The scale factor describes how all distances evolve in time, it is the essence of an evolving universe.

4.1 Distance and Coordinate-Distance

Central to a universe in which space evolves uniformly in time are two distinct notions, “distance” and “coordinate-distance”⁵. So central are they that they deserve to be introduced by a homely analogy.

Let us imagine that our earth is expanding, that its radius a is a function of the time t : $a = a(t)$. Let us denote the present epoch by t_0 . ($a(t_0) = 6371$ km). Let us suppose that we the observer are located at the North Pole. Now let us consider the distance of a place such as Datong from us. Let the co-latitude of Datong be denoted by χ ($=50^\circ$). Let us denote its present distance by r : we have

$$r = a(t_0) \cdot \chi \quad ,$$

and let us denote its distance at time t by D : we have

$$D = a(t) \cdot \chi \quad .$$

Now we introduce the scale factor $R(t)$ defined as

$$R(t) = a(t)/a(t_0) \quad , \quad (R(t_0) \equiv 1) \quad .$$

Then the previous expression can be re-written as

$$D = R(t) \cdot r \quad , \tag{13}$$

³Many authors (e.g., Felten et al. 1986) prefer to use the Schwarzschild radial coordinate, r_1 say, such that $2\pi r_1$ is the comoving circumference (Misner et al. 1970, p.723), and replace the $(dr)^2$ in (12) by $(dr_1)^2/[1 - k(r_1/a_0)^2]$.

⁴The Cosmological Principle that the universe must look the same to all observers is meaningless if the universe is evolving, unless we qualify the “look the same” with “at a given time”. Then a given appearance or state of the universe as measured by, say, the rate of expansion and/or the mean density, must be associated with a particular instant of time, a particular clock *reading*. So in regard to time, cosmology has out-newtoned Newton: not only do all clocks run at the same rate, they actually give the same reading to a given event.

⁵“Distance” is what is known as “proper distance” in GR, and “coordinate-distance” is sometimes called “comoving distance”

and this is the basic relation between the “distance” D on one hand and the “coordinate distance” r and the (normalised) scale factor $R(t)$ on the other. The distance from the observer to any point fixed to the earth, D , depends on time through $R(t)$, and the value of D at some standard time at which $R = 1$, is the coordinate-distance of that point, r : each point fixed to the earth is associated with a fixed value of r . For Datong, $r = 6371 \times (50^\circ \text{ in radians}) = 5560 \text{ km}$; for Guadalcanal ($\chi = 100^\circ$), $r = 11120 \text{ km}$. Guadalcanal is twice as far as Datong *now*, and will always be so, however the earth expands (or shrinks).

The situation in the expanding universe is exactly the same. All we have to do is to identify the r and $R(t)$ in (13) with the r and $R(t)$ in the FRW metric (12), and use (13) to define a distance D in the universe. The distance, D , from us to a particular point in the evolving space, then, is a function of *two* variables, the universal time t through the universal function $R(t)$, and the coordinate-distance r of the particular point under discussion: $D = D(t, r)$. The particular point *may or may not* happen to be occupied by a galaxy, a quasar, a speck of dust, a source of radiation, or a passing photon: its distance is always equal to the product of its coordinate-distance and the universal scale factor.

At any given time t , D is simply proportional to r : just like Datong and Guadalcanal, if the distance of quasar A at the present epoch is 1 Gly (one billion light-years), and that of quasar B is 2 Gly, then B will always be, and has always been, twice as far as A, whatever the form $R(t)$ in the future or in the past.

For a particular point in space with coordinate-distance r , its velocity of recession ⁶, v^* , is given by the partial derivative of D with respect to t , at fixed r :

$$v^* = \dot{R}(t) \cdot r . \quad (14)$$

This is, in fact, the primitive and a more practically useful form of Hubble’s Law. The usual form of Hubble’s Law is

$$v^* = H(t) \cdot D , \quad (H(t) \equiv \dot{R}(t)/R(t)) \quad (15)$$

obtained by eliminating r between (14) and (13), and introducing the Hubble parameter $H(t)$ as shown.

4.2 Status of Hubble’s Law

A common misunderstanding surrounds Hubble’s Law. It is often thought that Hubble’s Law comes directly from Hubble’s observations of the galaxies. Not so. Hubble, of course, did not observe *directly* the velocities and distances of the galaxies. What he observed was their redshifts and apparent magnitudes and he found a correlation between the two. Then, interpreting the redshifts as Doppler shifts, and the apparent magnitudes as a distance effect, the observed linear regression of redshift on apparent magnitude is converted into a linear relation between velocity of recession and distance, and the latter has come to be popularly known as Hubble’s Law. This law is thus based on an interpretation

⁶From now on we shall always talk of “recession”, with the understanding that it could be negative

of an empirical correlation, and as such, cannot be an exact law. (In fact, Hubble’s original data had a very large scatter (Hubble 1936), but whatever refinement later observations have brought about, an empirical relation can never imply an exact relation). Considered this way, it would be legitimate to say that, possibly, or even probably, the linearity between velocity and distance in the law is only approximate, valid only for small distances, and that as the distance increases indefinitely, the velocity of recession may approach the speed of light only asymptotically. That would neatly reconcile the extrapolation of this empirically-based law with the requirement of SR that nothing can move faster than light. (And after Hiroshima who can doubt the truth of SR ?)

However, this is not what is understood as Hubble’s Law in theoretical cosmology. Modern cosmological models come directly from GR and the CP, and from the latter comes the basic relation (13), and hence, the distance-velocity relation (14) or (15). It is this *strictly linear*, observation-independent, relation between distance and velocity of recession (with, however, a time-dependent “constant” of proportionality, the Hubble parameter $H = H(t)$) that is now understood as Hubble’s Law. From this perspective the significance of Hubble’s and his successors’ observations of the galaxies is just to tell us that the sign of the present value of H or \dot{R} is positive, rather than zero or negative, and to tell us what that positive value is.⁷ According to the Hubble’s Law (14), then, the velocity of recession at any time t is strictly proportional to the coordinate-distance r . If the universe is spatially open ($k = -1$) or flat ($k = 0$), then r can be indefinitely large, and so can the velocity of recession. If space is closed ($k = +1$), then r and hence velocity of recession, has a finite upper bound, but for any closed model at all consistent with the observations, recession velocities greater than c will have existed since the “beginning” up to a certain far point in future. For details and numerical illustrations see Kiang (1997).

A Caveat on the term ‘superluminal’ Recession velocities greater than the speed of light are sometimes referred to as “superluminal” velocities. They should be carefully distinguished from the “superluminal velocities” (often associated with “tachyons”) discussed by experimental and theoretical physicists. To add to the confusion, “superluminal jets” are said to have been observed in many radio sources. In this last case, the “superluminality” is a purely apparent effect⁸.

The point in space where the recession velocity equals c is nothing special. That we should have thought it special is because we have been psychologically conditioned to think always in terms of SR. It is high time to recognize the

⁷For orientation purposes, a nice round number of the Hubble time, H_0^{-1} , is 20 Gyr (20 billion years). Latest estimates would put it some 30% lower.

⁸If a jet leaves a source at a speed v and at an angle ψ to the direction towards the observer, then its apparent velocity, v_{app} , defined as the ratio of its transverse linear separation (got from its observed angular separation and any assumed large distance) to the observed time-interval, is, $v_{\text{app}} = v \sin \psi / (1 - v \cos \psi)$. The derivation is entirely newtonian, assuming only a constant light speed. Then for a v under 1, we can get v_{app} greater than 1. It is often thought that in addition, ψ has to be small; this is not true, in fact, for a given v , v_{app} maximizes at $\psi = 45^\circ$. In the astronomical context, however, small ψ is necessary for the jet to be sufficiently boosted relativistically to become visible (Rees 1967).

unhelpful character of SR thinking when working on cosmology.

In SR, $v = c$ is associated with infinite redshift, synonymous with observable limit. Now, recognising that SR is not valid in cosmology, the question “Can we observe a galaxy that recedes at the light speed ?” can no longer be dismissed with a categorical “Of Course Not”: the question has become non-trivial and interesting. And entirely tractable, as it turned out.

Nor can the question be dismissed as meaningless, on a higher level, by saying velocity at a distant point is not defined in GR. Cosmolgy is not just GR, it is GR plus the CP, and thanks to the latter, finite distances, and velocities at distant points, all referring to particular instants of time, are all perfectly defined.

4.3 How Photons Move in Expanding Space

Let us go back to the analogy of the expanding earth. Let us now suppose there is a special race of ants constantly crawling on the surface of this earth. The ants are special in that they always crawl at the same *ground* speed κ , regardless of what the ground is doing. Now consider a certain instant t and a certain location specified by co-latitude χ , such that, at that instant t , that place is receding from the North Pole at speed κ : $\dot{a}(t)\chi = \kappa$. We now ask, “Suppose an ant starts there and then to crawl towards us at the North Pole, will it ever reach us ?” It is obvious that the answer must simply depend on the given form of $a(t)$, and can be found using elementary calculus.

But this is exactly the same as the “cosmological” question we are asking, once we identify the ants with photons and κ with c . So the answer to our question simply depends on the form of $R(t)$, and can be got just as easily.

Now we leave the analogy and work out the cosmological question in full accordance with GR. The equation of motion for a photon in GR is given by $ds = 0$. Now, we are only interested in photons moving in the radial directin, so the FWR metric (12) is simplified to read

$$(ds)^2 = (dt)^2 - R^2(t) \cdot (dr)^2 \quad (16)$$

and $ds = 0$ is simply,

$$dr = \mp dt/R(t) \quad (17)$$

with the minus sign for incoming photon and the plus sign for outgoing photon.

This equation is the basis for answering the question that started the present enquiry. It also provides definitions of horizon and redshift in cosmology. It turned out that the distance at which $v^* = 1$ has nothing to do with horizon, nor anything to do with infinite redshift.

4.3.1 Can we observe galaxies that recede fast than light ?

Starting with Eq. (17) for the incoming photon and the primitive form of Hubble’s Law (14), and using no more than elementary calculus, I was able to answer this question completely (Kiang 1997). The answer is as follows. For the steady-state model ($R(t) = \exp(t)$, $-\infty < t < +\infty$, in Hubble units ⁹),

⁹We use Hubble units when we express all time intervals in units of the Hubble time, H_0^{-1}

the answer is “No”. For all the three varieties ($k = 0, \pm 1$) of the big bang model, the answer is “Yes”; moreover, we can say, for example, that for the $k = 0$ “standard” model, all quasars we now observe having redshifts greater than 1.25 (we now know thousands of these) have the property that, at the time of emission of the photon that now reaches us, they were receding faster than c . More recently (Kiang 2003), I have generalised the answer to, “Yes, if the universal expansion started with a singularity; No, if it started infinitely slowly from a finite size including zero;—this happens when the model contains a cosmological constant at a certain critical value (Bondi 1960, p. 82, Felten and Isaacman 1986)”.

4.3.2 Horizons

Starting with Eq (17) with the plus sign, and integrating from t_{begin} ($=0$, or $-\infty$ as the case may be) to any given epoch t results in the (particle) horizon (size of the observable universe) at the time t (Kiang 1991). If the universe had a finite past, then the horizon is finite. And obviously, this finite horizon has nothing to do with the distance at which v^* assumes any particular value including 1.

Integrating the same equation from a given epoch t to t_{end} ($= +\infty$, or some finite value, as the case may be), results in the “event horizon” at t , which does not directly concern the present enquiry (Kiang 1997)¹⁰.

The important thing to note here is that horizon, or the limit of observability in the universe, is a certain simple function of $R(t)$. It has nothing to do with infinite redshift, as we shall see presently. That the two should be linked is another misleading hang-over from SR.

4.4 Cosmological Redshift. Doppler Redshift

The equation of the incoming photon, Eq. (17) with the minus sign, is again used to derive the formula for the redshift in cosmology. The formula is (see e.g., Bondi 1960, p.106),

$$1 + z = \frac{1}{R(t_{\text{em}})}, \quad (R(t_{\text{obs}}) \equiv 1), \quad (18)$$

where t_{em} is the time of emission of the photon, and t_{obs} is the time of its observation, identified with the present epoch t_0 . Thus, cosmological redshift has nothing to do any velocity; it simply depends on the value of the scale factor at the epoch of emission; it is completely different from Doppler redshift (whether classical or relativistic), which is above all else a function of the radial velocity of the source with respect to the observer.

Of course, for a given model, i.e., a known $R(t)$, it is easy to work out a relation between the redshift z and the velocity of recession of the source at a

and all distances in units of the Hubble radius, cH_0^{-1} . Hubble units are one example of $c = 1$ units, and the latter are a sub-set of “normalised units”. For a general account of “normalised units”, see Kiang (1987)

¹⁰Confusingly, researchers on formation of large scale structures now use yet a third “horizon”, which turns out to be just the Hubble radius at the current epoch, $c/H(t)$

specified moment, e.g., the time of emission t_{em} , or the time of observation (the present epoch), t_0 . For the simplest big bang model (the standard model), with $R(t) = (t/t_0)^{2/3}$, I derived (Kiang 1995),

$$\left. \begin{aligned} v^*(t_{\text{em}}) &= 2(\sqrt{1+z} - 1), \\ v^*(t_0) &= 2(1 - 1/\sqrt{1+z}). \end{aligned} \right\} \quad (19)$$

However, it could never be over-emphasised that these particular $v^* - z$ relations are altogether different in character from the relativistic Doppler redshift formula ¹¹,

$$1 + z = \sqrt{1+v}/\sqrt{1-v}. \quad (20)$$

The formal difference is that each $v^* - z$ relation is specific to a cosmological model and for a particular characteristic time, whereas the Doppler formula is quite general. But the fundamental difference lies between v and v^* in their meaning: the v in the Doppler formula refers to motion of objects *in* space which itself is tacitly assumed to be *fixed*, whereas the v^* in (19) refers to motion of objects that are merely being carried along by a space, which itself is *expanding*. Equation (13), expressing the expansion, is basic to cosmology; it is alien to Special Relativity. We would only be misleading ourselves if we apply consequences and deductions of SR, deductions such as “nothing can move faster than light”, or “ $v = c$ implies infinite redshift”, to an expanding universe.

4.4.1 The Redshift of Superluminal Jet.

Superluminal jets (cf. the *Caveat* in 4.2) offer an instance where we are forced to consider simultaneously the two types of redshift, cosmological and Doppler. The problem can be formulated as follows. A radio source with observed redshift z_{source} has an (apparently) superluminal jet, which is supposed to be due to the jet moving at velocity $v_{\text{jet}} \lesssim 1$ relative to the source at a small angle ψ towards us. What would the redshift of the jet (z_{jet}) be, if observable? The answer I found (Kiang 2003) is that z_{jet} is to be given by $1 + z_{\text{jet}} = (1 + z_{\text{source}})(1 + z_{\text{Doppler}})$, where the last factor is to be calculated from the relativistic Doppler formula for the given v and $\theta = 180^\circ - \psi$, shown in the footnote to Eq. (20).

4.5 Recession Velocity. Peculiar Velocity

The versatile Eq. (17) describes the time variation of the coordinate-distance r of the photon moving in the radial direction. What about the time rate of its (proper) distance D ? From the definition of D at (13), we have, immediately,

$$\frac{dD}{dt} = r \cdot \frac{dR}{dt} + R \cdot \frac{dr}{dt} \quad (21)$$

The first term on the right is, according to (14), just the recession velocity at the current point, v^* , and the second reduces to ∓ 1 on using (17). Hence the “total” velocity of the photon (relative to us), at time t , at distance D , is

$$v_{\text{photon}} = v^* \mp 1. \quad (22)$$

¹¹The formula (20) is for purely radial motion. For motion at angle θ to the positive radial direction, the formula is $1 + z = (1 + v \cos \theta)/\sqrt{1 - v^2}$.

This equation says that the local velocity of the photon (∓ 1) is simply compounded with the recession velocity v^* in the old newtonian, pre-relativity manner (Kiang 1997). And there is nothing remarkable about that, once we are rid of Special Relativity preconceptions.

Formula (22) is a particular case of the more general formula for the “total” velocity (Davis and Lineweaver, 2000),

$$v_{\text{total}} = v^* + v_{\text{peculiar}} , \quad (23)$$

particularised to the case $v_{\text{peculiar}} = \mp 1$.

Another particularization would be the case of a relativistic jet coming out of a distant radio source, where the recession velocity would be referring to the source and the peculiar velocity is the radial component of the relative velocity of the jet with respect to the source.

A third particularization would be the case of a “comoving source” where v_{peculiar} is identically zero, and the total velocity is just the recession velocity.

Recession velocity and peculiar velocity are different animals. The recession velocity should not be regarded as the property of a source; rather, it should be considered as the property of the point of space in question, whether that point happens to be occupied by a source, a passing photon, or nothing at all. The peculiar velocity, on the other hand, must have reference to a material object including photon. The recession velocity is not the kind of velocity considered in SR, hence it is not subject to any of the laws of SR; the peculiar velocity *is*, and so *is* subject to all its laws: it must never exceed the speed of light and it gives rise to Doppler shift (Cf. 4.4.1). The recession velocity is always in the radial direction, whereas the peculiar velocity can be in any direction;—what appears in Eq. (23) is its radial component. After detailing all these essential differences, we almost marvel at the simply way they combine to give a total velocity. The simplicity comes from the underlying simplicity of the expanding space, as expressed by the Hubble Law. The two velocities are additive simply because they are just the two terms in the total differential of a function of two variables.

5 Shadows of Special Relativity

After Hiroshima, who of us can doubt the truth of Special Relativity ? So we take it for granted that nothing can move faster than light, that $v = c$ means infinite mass, infinite redshift, some absolute physical barrier, etc.. Then Hubble’s Law comes along and seems to say that there *are* galaxies that recede faster than light. How can we square that with SR ? So we consider a series of ways out. These are the “leading questions” at various levels spelt out in the introductory paragraph of this essay.

One by one these escape routes are proved untenable. They are untenable because we shouldn’t have considered them in the first place. The recession velocities of galaxies are, as it were, outside the remit of SR, the expanding space

in cosmology is not the fixed space of SR, so why *shouldn't* there be contradictions between SR and cosmology ?

Once the shadows of SR are cast away, the following broad features of our universe emerge easily enough. The difficulty has not been the mathematics, but the psychology.

1. At any given time the recession velocity is strictly proportional to the distance and there are certainly galaxies that recede faster than light.

2. Galaxies receding faster than light are observable if the universal expansion started with a bang at a finite past epoch; they are not observable, if the expansion started infinitely slowly.

3. Galaxies that recede at the speed of light are NOT associated with infinite redshift. On the contrary, infinite redshift in cosmology is associated with the *time* of the bang, if it exists, when the size of the universe is zero.

4. Horizon in the sense of a boundary to the observable region exists if the universe started a finite time ago¹² and then its value depends only on the form of the scale factor and has nothing to do with redshift.

It has been a personal journey of discovery. On the way I have identified a number of common misconceptions about time and space. I have pinpointed the origin of the twin paradox in the usual form of the Lorentz transformation given in textbooks. I have recognized the role played by the Cosmological Principle in lifting cosmology out of the general indeterminateness of General Relativity by providing it first with a universal time, then with well-defined notions of finite distances at a given time, and of indefinitely large recession velocities (regardless of the speed of light) at indefinitely large distances.

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¹²This statement also applies to a static universe. Horizon is certainly *not* a privileged property of expanding universes; any universe, static or expanding, has a horizon, so long as it has a finite past. In fact, for a static universe, the horizon is simply ct

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